Period 4, Jan 23, 2024

Double-Angle Formulas

$$\sin 2\theta = 2 \sin \theta \cos \theta$$
$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$
$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$\sin(a+b) = \sin a \cos b + \sin b \cos a$

$\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta$

$= 2\sin\theta\cos\theta$

(cos, sin)

$(-, +)$

CHECK POINT 1 If $\sin \theta = \frac{4}{5}$ and θ lies in quadrant II, find the exact value of each of the following:

a. $\sin 2\theta$ **b.** $\cos 2\theta$ **c.** $\tan 2\theta$.

$2\sin\theta\cos\theta$

$2 \cdot \frac{4}{5} \cdot \frac{-3}{5} = \frac{-24}{25}$

$\cos^2\theta - \sin^2\theta$

$\left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$

$\frac{9}{25} - \frac{16}{25} = \frac{-7}{25}$

$\frac{\sin 2\theta}{\cos 2\theta}$

$\frac{\frac{-24}{25}}{\frac{-7}{25}}$

$\frac{-24}{\cancel{25}} \cdot \frac{\cancel{25}}{-7} = \frac{24}{7}$

$3\frac{3}{7}$

$\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{4}{5}\right)^2 + \cos^2\theta = 1$

$\frac{16}{25} + \cos^2\theta = \frac{25}{25} - \frac{16}{25}$

$-\frac{16}{25}$

$\sqrt{\cos^2\theta} = \sqrt{\frac{9}{25}}$

$\cos\theta = \pm\frac{3}{5}$

$\cos\theta = -\frac{3}{5}$

Three Forms of the Double-Angle Formula for cos 2θ

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta - (\sin^2\theta + \cos^2\theta)$

$2\cos^2\theta - \sin^2\theta - \cos^2\theta$

$\cos^2\theta - \sin^2\theta$

$\cos(2\theta)$

EXAMPLE 3 **Verifying an Identity**

Verify the identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. ← no sin

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(2\theta + \theta)$

$\cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$(\cos^2\theta - \sin^2\theta)(\cos\theta) - 2\sin\theta\cos\theta \cdot \sin\theta$

$\cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos\theta$

$\cos^3\theta - 3\cos\theta\sin^2\theta$

$\cos^3\theta - 3\cos\theta(1 - \cos^2\theta) \Rightarrow$

$\cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$4\cos^3\theta - 3\cos\theta$

Verify the following identity.

$$-\cos 2\theta = \frac{1 - \cot^2\theta}{1 + \cot^2\theta}$$

$$\frac{1 - \frac{\cos\theta}{\sin\theta}}{1 + \frac{\cos\theta}{\sin\theta}}$$

$$\frac{\frac{\sin^2\theta}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}}{\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}}$$

$$\frac{\frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta}}{\frac{\sin^2\theta + \cos^2\theta = 1}{\sin^2\theta}}$$

$$\frac{\sin^2\theta - \cos^2\theta}{\cancel{\sin^2\theta}} \cdot \frac{\cancel{\sin^2\theta}}{1}$$

$\sin^2\theta - \cos^2\theta = -(\cos^2\theta - \sin^2\theta) = -\cos 2\theta$

Power-Reducing Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

$$\frac{\sin^2\theta + \cos^2\theta - (\cos^2\theta - \sin^2\theta)}{2}$$

$$\frac{\sin^2\theta + \cancel{\cos^2\theta} - \cancel{\cos^2\theta} + \sin^2\theta}{2} = \frac{2\sin^2\theta}{2} = \sin^2\theta$$

Solution Our goal is to rewrite $\cos^4 x$ without powers of trigonometric functions greater than 1. To achieve this goal, we will apply the formula for $\cos^2\theta$ twice.

$\cos^4 x = (\cos^2 x)^2$

$$\left(\frac{1+\cos 2\theta}{2}\right)^2$$

$$\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)$$

$$\cos^2 2\theta = \frac{1+\cos 4\theta}{2}$$

$$\frac{1+\cos 2\theta + \cos 2\theta + \cos^2 2\theta}{4}$$

$$\frac{\frac{2}{2} + \frac{4\cos 2\theta}{2} + \frac{1+\cos 4\theta}{2}}{4} = \frac{\frac{2+4\cos 2\theta + 1 + \cos 4\theta}{2}}{\frac{4}{1}}$$

$$\frac{3 + 4\cos 2\theta + \cos 4\theta}{2} \cdot \frac{1}{4}$$

$$\frac{3 + 4\cos 2\theta + \cos 4\theta}{8}$$

Half-Angle Formulas

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

The $\pm$ symbol in each formula does not mean that there are two possible values for each function. Instead, the $\pm$ indicates that you must determine the sign of the trigonometric function, + or −, based on the quadrant in which the half-angle $\frac{\alpha}{2}$ lies.

EXAMPLE 5 **Using a Half-Angle Formula to Find an Exact Value**

Find the exact value of cos 112.5°.

$\pm$?

Quad II $(-,+)$

$\cos 225 = -\frac{\sqrt{2}}{2}$

$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1 - -\frac{\sqrt{2}}{2}}{2}}$

$\cos\frac{225}{2} = -\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}}$

$-\sqrt{\frac{\frac{2+\sqrt{2}}{2}}{\frac{2}{1}}}$

$-\sqrt{\frac{2+\sqrt{2}}{2}\cdot\frac{1}{2}} = -\sqrt{\frac{2+\sqrt{2}}{4}}$

CHECK POINT 5 Use $\cos 210° = -\frac{\sqrt{3}}{2}$ to find the exact value of cos 105°.

$\cos\frac{210}{2}$

$\cos\frac{210}{2} = \pm\sqrt{\frac{\cos 210 + 1}{2}}$

$\pm\sqrt{\frac{-\frac{\sqrt{3}}{2}+1}{2}}$

Verify the identity: $\tan\theta = \dfrac{1 - \cos 2\theta}{\sin 2\theta}$. $= \dfrac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} = \dfrac{\cancel{1}-\cancel{1}+2\sin^2\theta}{2\sin\theta\cos\theta}$

$\dfrac{\cancel{2}\cancel{\sin\theta}\sin\theta}{\cancel{2}\cancel{\sin\theta}\cos\theta} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

Half-Angle Formulas for Tangent

$$\tan\frac{\alpha}{2} = \frac{1 - \cos\alpha}{\sin\alpha}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1 + \cos\alpha}$$

Verify the identity: $\tan\dfrac{\alpha}{2} = \csc\alpha - \cot\alpha$.

Principal Trigonometric Identities

Sum and Difference Formulas

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \qquad \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Double-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$$

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

Power-Reducing Formulas

$$\sin^2\theta=\frac{1-\cos 2\theta}{2} \qquad \cos^2\theta=\frac{1+\cos 2\theta}{2} \qquad \tan^2\theta=\frac{1-\cos 2\theta}{1+\cos 2\theta}$$

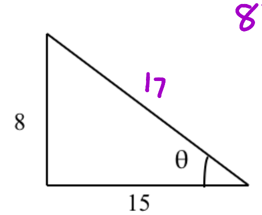
Half-Angle Formulas

$$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}=\frac{1-\cos\alpha}{\sin\alpha}=\frac{\sin\alpha}{1+\cos\alpha}$$

Use the figure to find the exact value of the following trigonometric function.

$$2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$



$8^2+15^2=17^2$

$\sin\theta=\frac{8}{17}$

$\cos\theta=\frac{15}{17}$

$$2\cdot\pm\sqrt{\frac{1-\cos\theta}{2}}\cdot\sqrt{\frac{1+\cos\theta}{2}}$$

$$2\pm\sqrt{\frac{1-\frac{15}{17}}{2}}\cdot\sqrt{\frac{1+\frac{15}{17}}{2}}=2\sqrt{\frac{(1-\frac{15}{17})(1+\frac{15}{17})}{2}}=2\sqrt{\frac{1-\frac{225}{289}}{2}}=\frac{289}{289}-\frac{225}{289}$$

$$2\sqrt{\frac{\frac{64}{289}}{\frac{2}{1}}}=2\sqrt{\frac{64}{17^2\cdot 2}}$$

$$2\cdot\frac{8\sqrt{2}}{17\sqrt{2}\sqrt{2}}=\frac{16\sqrt{2}}{34}$$

$17\cdot 2=34$

Quad IV $(+,-)$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

If **tan** $\alpha = -\dfrac{15}{8}$, $270^\circ < \alpha < 360^\circ$, then find the exact value of each of the following.

a. sin $\dfrac{\alpha}{2}$ **b. cos** $\dfrac{\alpha}{2}$ **c. tan** $\dfrac{\alpha}{2}$

(cos, sin)

Find the exact value of each of the following under the given conditions.

(−, +) (+, +)

$\tan\alpha = -\frac{4}{3}$, α lies in quadrant II, and $\cos\beta = \frac{5}{6}$, β lies in quadrant I

a. $\sin(\alpha+\beta)$ **b.** $\cos(\alpha+\beta)$ **c.** $\tan(\alpha+\beta)$

$\tan^2\alpha + 1 = \sec^2\alpha$

$\left(-\frac{4}{3}\right)^2 + 1$

$\frac{16}{9} + \frac{9}{9} = \sec^2\alpha$

$\sqrt{\frac{25}{9}} = \sqrt{\sec^2\alpha}$

$\pm\frac{5}{3} = \frac{1}{\cos\alpha}$

$-\frac{5}{3} = \frac{1}{\cos\alpha}$

$\cos\alpha = \frac{-3}{5}$

$\cos^2\alpha + \sin^2\alpha = 1$

$\left(\frac{-3}{5}\right)^2 + \sin^2\alpha = 1$

$\frac{9}{25} + \sin^2\alpha = \frac{25}{25} - \frac{9}{25}$

$-\frac{9}{25}$

$\sqrt{\sin^2\alpha} = \sqrt{\frac{16}{25}}$

$\sin\alpha = \pm\frac{4}{5}$

$\sin\alpha = +\frac{4}{5}$

$\cos^2\beta + \sin^2\beta = 1$

$\left(\frac{5}{6}\right)^2 + \sin^2\beta = 1$

$\frac{25}{36} + \sin^2\beta = \frac{36}{36} - \frac{25}{36}$

$-\frac{25}{36}$

$\sqrt{\sin^2\beta} = \sqrt{\frac{11}{36}}$

$\sin\beta = \pm\frac{\sqrt{11}}{6}$

$\sin\beta = \frac{\sqrt{11}}{6}$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$\frac{4}{5}\cdot\frac{5}{6} + \frac{\sqrt{11}}{6}\cdot\frac{-3}{5}$

$\frac{20}{30} + \frac{-3\sqrt{11}}{30} = \frac{20-3\sqrt{11}}{30}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\frac{-3}{5}\cdot\frac{5}{6} - \frac{4}{5}\cdot\frac{\sqrt{11}}{6}$

$\frac{-15}{30} - \frac{4\sqrt{11}}{30} = \frac{-15-4\sqrt{11}}{30}$

$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha\beta)} = \frac{\frac{20-3\sqrt{11}}{30}}{\frac{-15-4\sqrt{11}}{30}}$

$\frac{20-3\sqrt{11}}{\cancel{30}} \cdot \frac{\cancel{30}}{-15-4\sqrt{11}} = \frac{(20-3\sqrt{11})(-15+4\sqrt{11})}{(-15-4\sqrt{11})(-15+4\sqrt{11})}$

$\frac{-300+80\sqrt{11}+45\sqrt{11}-12\cdot 11}{225-\cancel{60\sqrt{11}}+\cancel{60\sqrt{11}}-16\cdot 11} = \frac{-300+125\sqrt{11}-132}{225-176} = \frac{-432+125\sqrt{11}}{49}$

Find the exact value of the expressions **cos** $(\alpha+\beta)$, **sin** $(\alpha+\beta)$ and **tan** $(\alpha+\beta)$ under the following conditions:

$(+,-)$ $(-,-)$

cos $(\alpha) = \frac{15}{17}$, α lies in quadrant IV, and **sin** $(\beta) = \frac{-2}{5}$, β lies in quadrant III.

$\cos^2\alpha + \sin^2\alpha = 1$

$\left(\frac{15}{17}\right)^2 + \sin^2\alpha = 1$

$\frac{225}{289} + \sin^2\alpha = \frac{289}{289} - \frac{225}{289}$

$-\frac{225}{289}$

$\sqrt{\sin^2\alpha} = \sqrt{\frac{64}{289}}$

$\sin\alpha = \pm\frac{8}{17}$

$\sin\alpha = -\frac{8}{17}$

$\cos^2\beta + \sin^2\beta = 1$

$\cos^2\beta + \left(-\frac{2}{5}\right)^2 = 1$

$\cos^2\beta + \frac{4}{25} = \frac{25}{25}$

$-\frac{4}{25}$ $-\frac{4}{25}$

$\sqrt{\cos^2\beta} = \sqrt{\frac{21}{25}}$

$\cos\beta = \frac{\pm\sqrt{21}}{5}$

$\cos\beta = -\frac{\sqrt{21}}{5}$

$(\sin, \cos)$

Find the exact value of each of the following under the given conditions.

a. cos $(\alpha+\beta)$ **b. sin** $(\alpha+\beta)$ **c. tan** $(\alpha+\beta)$

Quad III Quad 4 $(+,-)$

tan $\alpha = \frac{3}{4}$, $\pi < \alpha < \frac{3\pi}{2}$, and **cos** $\beta = \frac{2}{\sqrt{13}}$, $\frac{3\pi}{2} < \beta < 2\pi$

$(-,-)$

Use a sketch to find the exact value of the following expression.

$\sec\left[\sin^{-1}\left(-\frac{8}{9}\right)\right] = \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{\sqrt{17}}{9}} = \frac{9}{\sqrt{17}}\frac{\sqrt{17}}{\sqrt{17}}$

$\sin^{-1}\left(-\frac{8}{9}\right) = \theta$

$\sin\theta = -\frac{8}{9}$

$\sin^2\theta + \cos^2\theta = 1$

$\left(-\frac{8}{9}\right)^2 + \cos^2\theta = 1$

Range for θ

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ Quad I, IV

$\cos\theta = +$

$\sec\left[\sin^{-1}\left(-\frac{8}{9}\right)\right] = \frac{9\sqrt{17}}{17}$

(Use integers or fractions for any numbers in the expression. Type an exact answer, using radicals as needed. Rationalize all denominators.)

$\frac{64}{81} + \cos^2\theta = \frac{81}{81} - \frac{64}{81}$

$-64/81$

$\sqrt{\cos^2\theta} = \sqrt{\frac{17}{81}}$

$\cos\theta = \pm\frac{\sqrt{17}}{9}$

$\cos\theta = \frac{\sqrt{17}}{9}$

Verify the identity.

$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\beta - \sin^2\alpha$

$(\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = \cos^2\beta\cos^2\alpha$

$+ \cos\alpha\cos\beta\sin\alpha\sin\beta$

$- \cos\alpha\cos\beta\sin\alpha\sin\beta$

$- \sin^2\alpha\sin^2\beta$

Write the left side of the identity using the sum and difference formula.

$(\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

(Type the terms of your expression in the same order as they appear in the original expression.)

Use the distributive property to simplify the expression from the previous step.

$\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta$ (Simplify your answer.)

$\cos^2\beta\cos^2\alpha - \sin^2\beta\sin^2\alpha$

$\cos^2\beta\cos^2\alpha - (1-\cos^2\beta)\sin^2\alpha$

Substitute $1-\cos^2\beta$ for $\sin^2\beta$ in the expression from the previous step and distribute.

$\cos^2\alpha\cos^2\beta - \sin^2\alpha + \sin^2\alpha\cos^2\beta$

$\cos^2\beta\cos^2\alpha - \sin^2\alpha + \sin^2\alpha\cos^2\beta$

Regroup and factor the expression from the previous step. Do not apply any trigonometric identities yet.

$(\cos^2\alpha + \sin^2\alpha)\cos^2\beta - \sin^2\alpha$

$\cos^2\beta(\cos^2\alpha + \sin^2\alpha) - \sin^2\alpha$

$\cos^2\beta \cdot 1 - \sin^2\alpha$

Write the trigonometric expression as an algebraic expression containing x and y. Assume that x and y are positive and in the domain of the given inverse trigonometric function.

$\cos\left[\sin^{-1}x - \cos^{-1}y\right]$ $= \cos(a+b) = \cos a \cos b - \sin a \sin b$

$\sqrt{1-x^2} \cdot y \quad - x \cdot \sqrt{1-y^2}$

$\cos\left[\sin^{-1}x - \cos^{-1}y\right] = y\sqrt{1-x^2} + x\sqrt{1-y^2}$

$\sin^{-1}x = a$

$\sin a = x$

$\cos^{-1}y = b$

$\cos b = y$

$\sin^2 a + \cos^2 a = 1$

$x^2 + \cos^2 a = 1$

$\cos^2 a = 1 - x^2$

$\cos a = \sqrt{1-x^2}$

$\sin^2 b + \cos^2 b = 1$

$\sin^2 b + y^2 = 1$

$\sin^2 b = 1 - y^2$

$\sin b = \sqrt{1-y^2}$